Exercise 4

1. The probability that an insect lays r eggs is Poisson with mean λ, and the probability that an egg develops is p. Assuming independence between the eggs, find the probability that k eggs survive, $k = 0, 1, 2, \ldots$. (We have a joint distribution for (K, R) and want the marginal probability function for K.)

2. Suppose that the number of insurance claims that a particular policyholder makes in one year has a Poisson distribution with mean Λ, and that over the large population of policyholders Λ has a Gamma distribution. Find the probability of x claims in one year $x = 0, 1, \ldots$ by a policyholder chosen at random from the population of policyholders. (There is a joint distribution for (X, Λ), and we require the marginal distribution of X.)

3. Suppose that a bus is X minutes late, where X has an exponential distribution with mean $1/\Theta$, where Θ varies randomly with the density of the traffic according to the Gamma density function

$$f(\theta) = \lambda^\alpha \theta^{\alpha-1} e^{-\lambda\theta} / \Gamma(\alpha).$$

   What is the density function of X over all values of Θ? (Write down the joint density function for (X, Θ) and integrate out over values of Θ to get the marginal density function of X.)

4. If X and Y have joint distribution given by

$$f_{X,Y}(x, y) = \begin{cases} 2 & 0 < x < y < 1 \\ 0 & \text{elsewhere} \end{cases}$$

   find the covariance of X and Y, and the conditional density function of Y given $X = x$.

5. Let the joint density function for X, Y be

$$f_{X,Y}(x, y) = \begin{cases} 8xy & 0 < y < x < 1 \\ 0 & \text{elsewhere} \end{cases}$$

   (a) Find $E[Y|X = x]$.

   (b) Find $\text{Var}[Y|X = x]$.

   (c) Find $E[XY|X = x]$, and hence the covariance of X and Y.

6. Suppose that the joint density function of (X, Y) is

$$f_{X,Y}(x, y) = \begin{cases} 1 - a(1 - 2x)(1 - 2y) & 0 < x < 1, 0 < y < 1 \\ 0 & \text{elsewhere} \end{cases}$$

   where $-1 < a < 1$.

   Prove that X and Y are independent if and only if X and Y are uncorrelated.

7. Find the density function of the sum of two independent exponential random variables with the same mean. Do the same for the case where the means are different.

8. Let
$$Y = \begin{cases} \sum_{i=1}^{N} X_i & \text{if } N \geq 1 \\ 0 & \text{if } N = 0 \end{cases}$$
where $\{X_i\}$, $i = 1, 2, \dots$ is a sequence of iid random variables and N is a discrete random variable taking values $0, 1, \dots$ independent of $\{X_i\}$. Let $K_Y(t), K_N(t), K_X(t)$ denote the cumulant generating functions of Y, N and X_i respectively. Prove that
$$K_Y(t) = K_N[K_X(t)].$$
Furthermore, suppose that $P(N = i) = p^{i-1}(1-p)$ for $i = 1, 2, \dots$ and the density of X_i is given by $\alpha^2 x e^{-\alpha x}$ for $x > 0$. Prove that Y has density function
$$\frac{\alpha(1-p)}{2\sqrt{p}} \left(e^{-\alpha(1-\sqrt{p})y} - e^{-\alpha(1+\sqrt{p})y} \right),$$
for $y > 0$.

9. Suppose that X and Y are bivariate normally distributed with zero means, unit variances and correlation coefficient ρ. Show that the correlation coefficient between X^2 and Y^2 is ρ^2. (Use bivariate cumulants.)

10. Find the joint moment generating function of X and X^2 where X is a standard normal random variable.

11. Find the moment generating function for the product of two independent standard normal random variables, X and Y, by first finding the conditional moment generating function of XY for a fixed value of X. Hence or otherwise show that if U, V, X and Y are independent standard normal random variables, then $UV + XY$ has a Laplace distribution.

12. If X and Y are iid exponential random variables, find the joint density of $U = X/Y$ and $V = X+Y$.

13. Suppose that (R, Θ) are the polar coordinates of the random variables (X, Y) and that R has the density function $f_R(r)$ and is independent of Θ which has a uniform distribution on $(0, 2\pi)$. Find an expression for the joint density of (X, Y) in terms of f_R. Interpret your result geometrically.

14. If (X, Y) have some joint distribution, then one might try to define the concept of 'Y stochastically bigger than X' by the property $F_X(z) > F_Y(z)$ for all real z. Show that

   (a) If $F_X(z) > F_Y(z)$ for all z, then $E[Y] > E[X]$. You may assume X and Y are continuous random variables.

   (b) If $F_X(z) > F_Y(z)$ for all z, then $P[Y > X] > 0$.

   (c) If X and Y are independent continuous random variables, and $F_X(z) > F_Y(z)$ for all z, then $P[Y > X] > 0.5$.

